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C^1 Cubic Trigonometric Spline with a Shape Parameter for Positive Shape Preservation

Munir, N. A. A. A., Hadi, N. A.*, and Nasir, M. A. S.

Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA (Cawangan Shah Alam), Malaysia

E-mail: normi@fskm.uitm.edu.my
**Corresponding author*

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Abstract

This paper presents a new construction of C^1 cubic trigonometric spline interpolation. Instead of repositioning control points, a shape parameter is introduced in the spline to control the shape and behaviour of the curves. The built basis functions fulfil all the geometric properties of the standard cubic Bezier curve, and the proof is included in this paper. Then, the interpolation of the spline is illustrated using suitable parameter values. Every curve segment comprises four successive control points with a cubic trigonometric spline that carries out all the curve properties. The result showed effective approximation since the developed C^1 cubic trigonometric spline produced a smooth and pleasant interpolating curve while preserving the positive data features. The flexibility of the developed spline is compared with the other two existing works: b-spline and bezier-like curves. The analysis shows that the proposed spline gives greater flexibility since it has a broader parameter value range. Therefore, this helps the spline interpolation build opened and closed curves, as incorporated in the paper.

Keywords: continuity; interpolation; shape parameters; trigonometric spline.

1 Introduction

Computer Aided Geometric Design (CAGD) deals with many industrial areas, specifically in automotive, aerospace and robotics to fulfil the demands and their issues in constructing and designing shapes. Generating smooth curves and surfaces is an essential aspect of CAGD. Relevant basis functions with shape parameters are needed in preserving the shape of the curve and surface. The purpose of this paper is to generalize piecewise cubic trigonometric curves that achieve the required conditions for C^1 continuity.

2 Related Works

Due to some limitations in constructing free form curves using conventional methods based on polynomial, many researchers over the years proposed trigonometric splines based on trigonometric polynomials. Munir et al. [16] generated a new quadratic trigonometric beta spline with one shape parameter in preserving characteristics of positive data. In constructing smooth spline, relevant shape parameter values are assigned in achieving geometric continuity, G^1. A new kind of quasi cubic trigonometric Bernstein basis function with two shape parameters is constructed by [4] that produced positivity which achieves C^2 continuity. Han [8] presented practical piecewise cubic trigonometric polynomials with a shape parameter analogous to the cubic B-spline. Knot vector and value of shape parameter showed their importance in the determination of continuity throughout this paper. A shape parameter does not allow adjusting as much as shape control. Hence, a cubic trigonometric Bezier curve analogous to the cubic Bezier curve, with two shape parameters, is introduced by [7]. By changing the values of shape parameters, the shape of the curve is adjustable without repositioning the control polygon. This problem has been agreed upon by [20] who presents a rational cubic trigonometric Bezier curve with four shape parameters. A smooth joint of rational trigonometric curves with C^2 continuity is achieved when altering the shape parameters and weights. Hajji et al. [5] performed curves and surfaces using a new cubic Hermite trigonometric spline with shape parameters. Various values of shape parameters are tested to know the effects on the freeform curves and surfaces. They showed the curve and surface able to achieve C^3 continuous for specific values of shape parameters. Peng and Zhu [17] presented a new class of trigonometric Bezier basis functions with six shape parameters. The presence of six shape parameters gives advantages in adjusting and generating a triangular Bezier patch.

Abdul Karim [1] constructed a new rational cubic spline with three shape parameters which two of which are free, and another one is fixed to achieve shape-preserving. The proposed scheme succeeds to generate a positive interpolating curve elsewhere. Munir et al. [15] developed G^1 cubic trigonometric spline with the presence of two shape parameters β_1 and β_2 in fitting positive data that results in an effective realistic approximation since it manages to retain features of the real data. A year later, [14] continued the work by generating schemes that interpolate curves and data that not only lie at the position $y = 0$ axis but also should be above, below or between line or constrained $f_i = mx_i + c$. Han et al. [6] analyzed the problem of cubic trigonometric polynomial curves and state the conditions of the curve when having the loops, cusps and inflection points. The shape parameter competencies in controlling and modifying the curve shape are also discussed in the paper. Next, a new class of cubic trigonometric Bezier curve with a shape parameter similar to the cubic polynomial Bezier curve is presented by [11] and [18]. Both papers agreed that their new trigonometric curve has the capability to represent and deal precisely with transcendental curves, including circular arcs, cylinders and conics. A method to decide the values of shape parameter is proposed to get a smooth $\alpha - \beta$- spline curve, [9]. The $\alpha - \beta$-spline curves are also able to perform ellipses and parabolas under certain proper circumstances. Maqsood et al.[13] developed two shape parameters in trigonometric basis functions that are able to satisfy the parametric continuity (C^0, C^1, C^2 and C^3) and geometric continuity (G^0, G^1 and G^2) conditions. By modifying the two shape parameters, their proposed scheme can generate trigonometric surfaces over triangles.

In this paper, a new basis function of the cubic trigonometric spline is developed with a shape parameter. The generated spline is able to produce a smooth curve that meets the conditions needed for C^1 continuity. The work of the paper is arranged as follows. In section 2, the shape parameter with cubic trigonometric basis functions are described, and the property of the basis functions is discussed. Shape parameters are tested and analysed in providing intuitive control on the shape of the curve in section 3. Opened and closed curves are also presented in this section. The C^1 continuous condition for joining two constructed curves interpolation is presented in section 4. In section 5, the impact of shape parameter to preserve the positivity data and the comparison of the flexibility of presented trigonometric spline with two different proposed splines are discussed. Lastly, the paper is concluded, and future works are proposed.

3 Cubic Trigonometric Spline Basis Functions

The developed cubic trigonometric spline basis functions are given in the following definition,

Definition 3.1. *Cubic trigonometric basis functions with a shape parameter, m where $m \in [-2, 1]$ are defined for $t \in [0, 1]$:*

$$\begin{aligned}
B_0(t) &= (1 - \sin\frac{\pi}{2}t)^2(1 - m\sin\frac{\pi}{2}t)\\
B_1(t) &= m(\sin\frac{\pi}{2}t)(1 - \sin\frac{\pi}{2}t)^2 - 2(\sin\frac{\pi}{2}t)(\sin\frac{\pi}{2}t - 1)\\
B_2(t) &= m(\cos\frac{\pi}{2}t)(1 - \cos\frac{\pi}{2}t)^2 - 2(\cos\frac{\pi}{2}t)(\cos\frac{\pi}{2}t - 1)\\
B_3(t) &= (1 - \cos\frac{\pi}{2}t)^2(1 - m\cos\frac{\pi}{2}t)
\end{aligned} \tag{1}$$

with B_i as the basis functions. The range of shape parameter m is decided to fulfil the properties of the basis functions as in Theorem 3.1.

Theorem 3.1. *The basis functions in (1) satisfy all the properties of trigonometric spline as given below:*

i) *Non-negativity:*

$$B_i(t) \geq 0, i = 0, 1, 2, 3.$$

ii) *Partition of unity:*

$$\sum_{i=0}^{3} B_i(t) = 1, t \in [0, 1].$$

iii) *Monotonicity :*

According to the value of the parameter, m, $B_0(t)$ is monotonically decreasing, and $B_3(t)$ is monotonically increasing.

iv) *Symmetry:*

$$B_i(t, m) = B_{3-i}(1 - \frac{\pi}{2}t; m), i = 0, 1, 2, 3.$$

Proof. Example text of a proof.

i) For $t \in [0,1]$, and $m \in [-2,1]$, terms in $B_0(t)$ and $B_3(t)$ are

$$\begin{aligned} &(1-\sin\frac{\pi}{2}t)^2 \geq 0, && (1-m\sin\frac{\pi}{2}t), && \text{for } m \leq 1,\\ &(1-\cos\frac{\pi}{2}t)^2 \geq 0, && (1-m\cos\frac{\pi}{2}t), && \text{for } m \leq 1. \end{aligned}$$

Therefore, $B_0(t)$ and $B_3(t)$ are always non-negative for $m \leq 1$. For $B_1(t)$, the non-negativity is when

$$m(\sin\frac{\pi}{2}t)(1-\sin\frac{\pi}{2}t)^2 - 2(\sin\frac{\pi}{2}t)(\sin\frac{\pi}{2}t-1) \geq 0,$$

where,

$$m \geq \frac{-2}{(1-\sin\frac{\pi}{2}t)}.$$

Therefore, the non-negativity of $B_1(t)$ is when, $m \geq -2$. Similarly for $B_2(t)$, the non-negativity is written as,

$$m(\cos\frac{\pi}{2}t)(1-\cos\frac{\pi}{2}t)^2 - 2(\cos\frac{\pi}{2}t)(\cos\frac{\pi}{2}t-1) \geq 0,$$

where $m \geq \frac{-2}{(1-\cos\frac{\pi}{2}t)}$ and $m \geq -2$. Therefore, $B_1(t)$ and $B_2(t)$ are always non-negative for $m \geq -2$. Hence,

$$B_i(t) \geq 0, \quad i = 0,1,2,3\text{for} \quad m \in [-2,1].$$

ii) For all $t \in [0,1], m \in [-2,1]$,

$$B_0(t)+B_1(t)+B_2(t)+B_3(t) = 2-(\sin^2\frac{\pi}{2}t+\cos^2\frac{\pi}{2}t) = 1.$$

Therefore, $\sum_{i=0}^{3} B_i(t) = 1$ is proved.

iii) For $t_0, t_1 \in [0,1]$, such that $t_0 \leq t_1, B_0(t_0) \geq B_0(t_1)$ which shows that $B_0(t)$ is monotonically decreasing.

For $t_0, t_1 \in [0,1]$, such that $t_0 \leq t_1$,$B_3(t_0) \geq B_3(t_1)$ which shows that $B_3(t)$ is monotonically increasing.

iv) For $i = 2$,

$$\begin{aligned} B_2(t,m) &= m(\cos\frac{\pi}{2}t)(1-\cos\frac{\pi}{2}t)^2 - 2(\cos\frac{\pi}{2}t)(\cos\frac{\pi}{2}t-1)\\ &= m(\sin(1-\frac{\pi}{2}t))(1-\sin(1-\frac{\pi}{2}t))^2 - 2(\sin(1-\sin\frac{\pi}{2}t))(\sin(1-\frac{\pi}{2}t)-1)\\ &= B_1(1-\frac{\pi}{2}t;m). \end{aligned}$$

For $i = 3$,

$$\begin{aligned} B_3(t,m) &= (1-\cos\frac{\pi}{2}t)^2(1-m\cos\frac{\pi}{2}t)\\ &= B_0(1-\frac{\pi}{2}t;m). \end{aligned}$$

$\square$

In Figure 1, the basis functions of cubic trigonometric spline are fitted with various choice of shape parameter m, where the value of m must be within the range $[-2, 1]$.

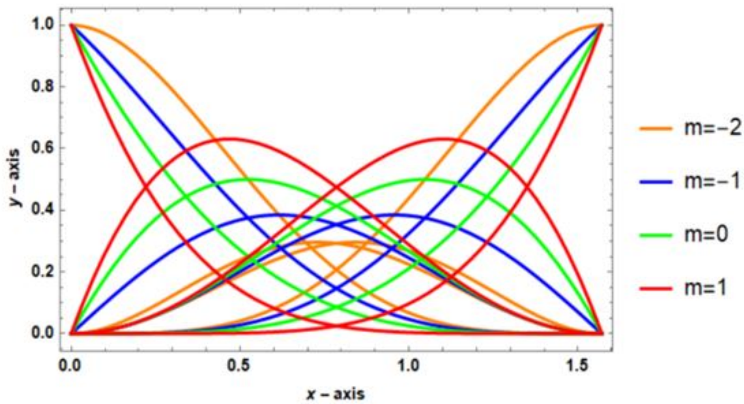


Figure 1: The cubic trigonometric basis functions for different value of m.

The role of the shape parameter will be discussed in the next section.

4 Shape Control of Cubic Trigonometric Ppline

The role of the shape parameter, m for interpolating the shape of the curves has been focused on in this section. The curve equation, $P(u)$ is given as,

$$P(u) = \sum_{i=0}^{3} B_i(u)Q_i, \tag{2}$$

where $B_i(u)$ is the basis function as in (1) and Q_i is the i^{th} control point. The fitted basis function with general control points is given in Figure 2.

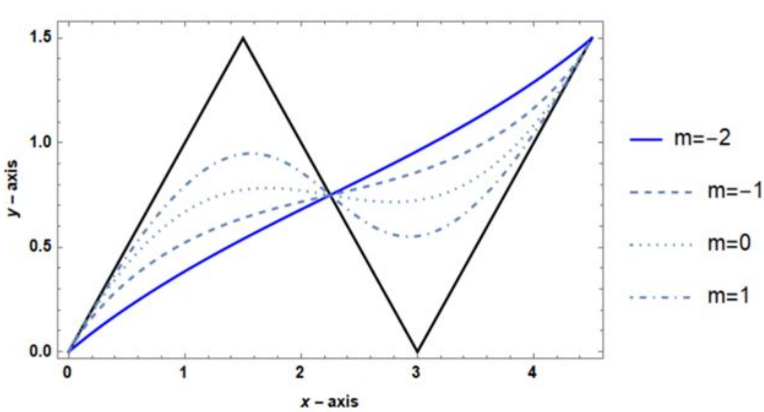


Figure 2: The effects of the shape parameter on the shape of cubic trigonometric basis functions.

In Figure 2, four consecutive control points $Q_i, (i = 0, 1, 2, 3)$ are randomly chosen, which are $Q_0 = (0, 0), Q_1 = (1.5, 1.5), Q_2 = (3, 0)$ and $Q_3 = (4.5, 1.5)$ in generating the cubic trigonometric spline curve. The curves design depends on the values of the shape parameter, m. The fitted curve approaches the control polygon when the shape parameter values are increasing. The fitted curves in Figure 2 can be extended to interpolate a particular shape, such as four-petal flowers, as shown in Figure 3 and 4.

Figure 3 and 4 show closed and opened curves generated by cubic trigonometric spline, respectively with various shape parameter values. The shape parameter's presence helps us represent the desired shape of curves without modifying the control points position. The shape parameter has local control on the interpolation without affecting the other curves.

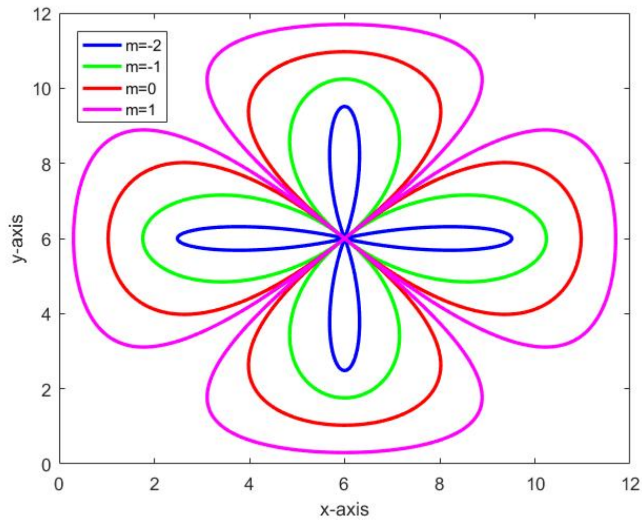


Figure 3: Closed curves.

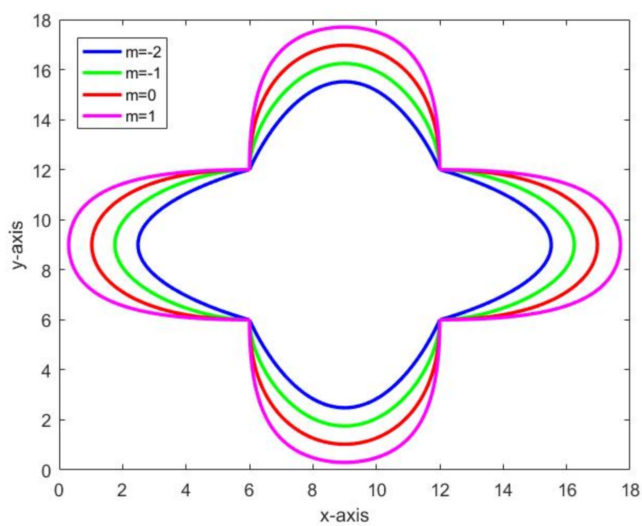


Figure 4: Opened curves.

In the figures, the control points are user-decided based on the required shape. The following section will discuss several real positive datasets with C^1 continuity.

5 C^1 Cubic Trigonometric Spline Interpolation

Let $(x_i, f_i), i = 1, 2, ..., n$ be the given set of data points defined over the subinterval $[x_i, x_{i+1}]$, where $x_0 < x_1 < ... < x_n$ and f_i is the function values. Based on (2), the piecewise cubic trigonometric function, $P_i(x)$ with one shape parameter m, is defined as:

$$\begin{aligned} P_i(x) = (1-\sin\frac{\pi}{2}t)^2(1-m\sin\frac{\pi}{2}t)V_i + m(\sin\frac{\pi}{2}t)(1-\sin\frac{\pi}{2}t)^2 - 2(\sin\frac{\pi}{2}t)(\sin\frac{\pi}{2}t-1)W_i \\ + m(\cos\frac{\pi}{2}t)(1-\cos\frac{\pi}{2}t)^2 - 2(\cos\frac{\pi}{2}t)(\cos\frac{\pi}{2}t-1)X_i + (1-\cos\frac{\pi}{2}t)^2(1-m\cos\frac{\pi}{2}t)Y_i. \end{aligned} \tag{3}$$

To find the order derivatives d_i at knots x_i, the formula is borrowed from [10]:

$$\begin{aligned} d_0 &= \Delta_0 + (\Delta_0 - \Delta_1)h_0(h_0+h_1)^{-1}, \\ d_n &= \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2})h_{n-1}(h_{n-1}+h_{n-2})^{-1}, \\ d_i &= (h_{i-1}\Delta_i + h_i\Delta_{i-1})(h_{i-1}+h_i)^{-1}; \;\; i = 0, 1, 2, ..., n-1 \end{aligned} \tag{4}$$

where $\Delta_i = (f_{i+1} - f_i)/h_i, \;\; h_i = x_{i+1} - x_i, \;\; t = (x - x_i)/h_i, \;\; i = 0, 1, 2, n-1$, and d_i is the derivatives by the Arithmetic Mean Values.

A C^1 interpolating continuity is achieved by cubic trigonometric spline by the following conditions,

$$\begin{aligned} P(x_i) &= f_i, \qquad P(x_{i+1}) = f_{i+1}, \\ P'(x_i) &= d_i, \qquad P'(x_{i+1}) = d_{i+1}, \end{aligned} \tag{5}$$

which implies the control points in (3) as,

$$\begin{aligned} V_i &= f_i \\ W_i &= \frac{2h_id_i}{\pi(m+2)} + f_i \\ X_i &= \frac{-2h_id_{i+1}}{\pi(m+2)} + f_{i+1} \\ Y_i &= f_{i+1}. \end{aligned} \tag{6}$$

Based on the conditions, the new interval of the shape parameter is obtained, where $m \in (-2, 1]$. The numerical examples are shown in the next section.

6 Numerical Example

In this section, first, the influence of the shape parameter in preserving the positivity of positive data is presented. Then, a comparison of the developed C^1 cubic trigonometric spline with two other different proposed splines is presented.

Example 1: Table 1 shows a positive data set taken from [3] where $x_0 < x_1 < ... < x_n$ and $f_0 > 0, f_1 > 0, ..., f_n > 0$. Choosing appropriate values of shape parameter, m is important to preserve the features of the data in the subinterval if $P_i(x) > 0, i = 0, 1, 2, 3, ..., n$.

Table 1: A positive data set by [3].

i	**0**	**1**	**2**	**3**	**4**
x_i	1	2	3	4	5
f_i	3	6	5	8	1

The positive data in Table 1 is interpolated in Figure 5 and 6 by C^1 cubic trigonometric spline with random values of the free parameter, m. The numerical results showed in Table 2 and 3.

Table 2: Tested value 1.

i	**0**	**1**	**2**	**3**	**4**
m	1	2	3	4	5
d_i	3	6	5	8	1
Δ_i	3	-1	3	-7	-

Table 3: Tested value 2.

i	**0**	**1**	**2**	**3**	**4**
m	0	1	0	-1	-
d_i	5	1	1	-2	0
Δ_i	3	-1	3	-7	-

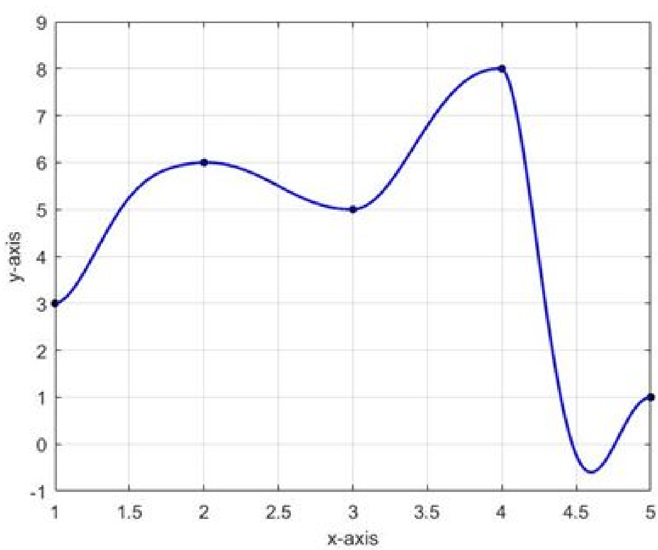


Figure 5: Non-positive curve.

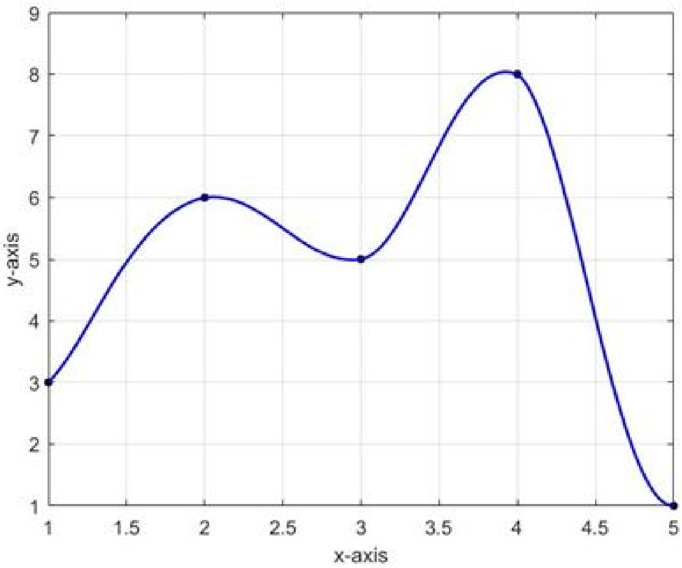


Figure 6: Positivity preserving curve.

In Figure 5, C^1 cubic trigonometric functions fail to preserve the characteristics of positive data when random values of the shape parameter, m are chosen as shown in the numerical results in Table 2. The C^1 cubic trigonometric function is then tested with another arbitrary parameter value, m in Table 3. The result is shown in Figure 6. The curve produced preserves the positivity of the data. The selection of appropriate values of the shape parameter is important in getting a smooth curve and a curve that preserves the hereditary of the data.

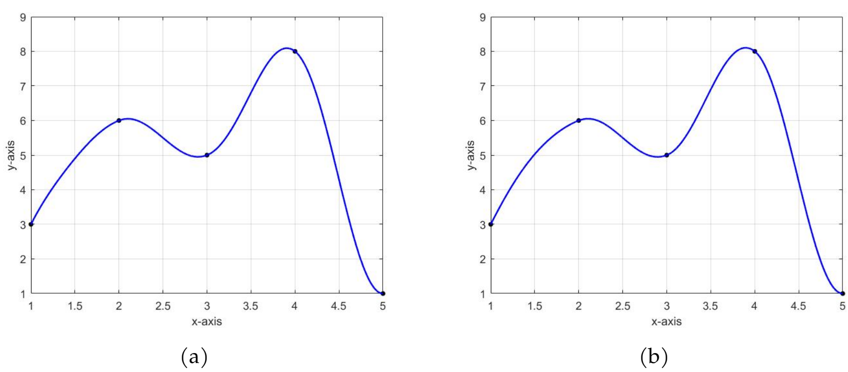


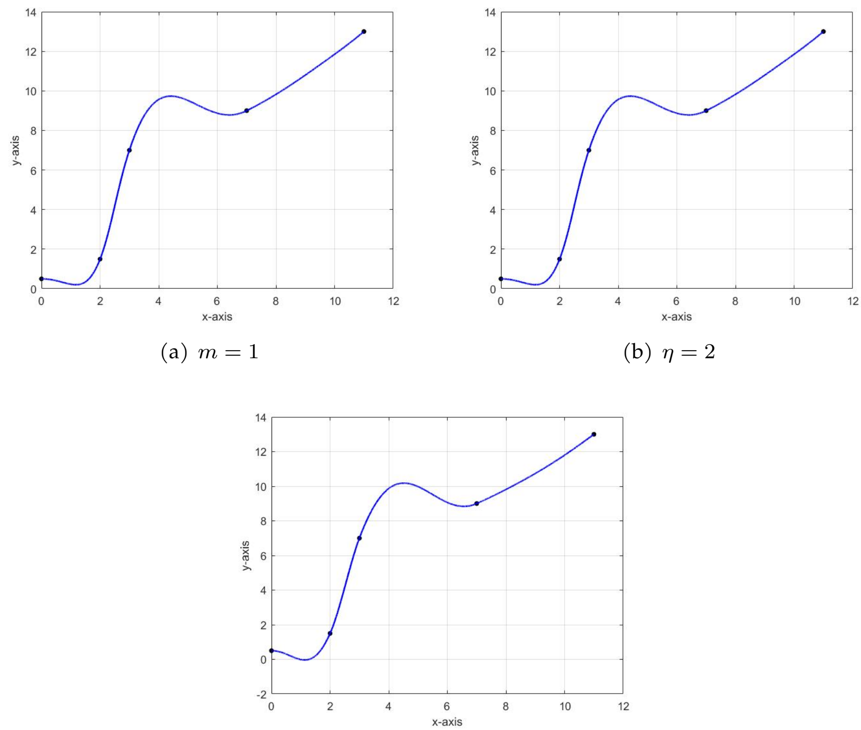
Figure 7: Positivity preserving curve using: (**a**) Cubic trigonometric B-spline. (**b**) Trigonometric cubic Bezier-like curve.

Cubic trigonometric B-spline by [12] in 7(a) which have a shape parameter, $\eta = (1/2, 2]$ and trigonometric cubic Bezier-like curve by [19] in 7(b) with the parameter $\alpha = (0, 2)$ are generated using positive data set in Table 1. By choosing one of the values of shape parameter within the range given, both curves in Figure 7 preserve the positivity of the data.

Example 2: A positive data set in Table 4 is borrowed from [2]. The data set is interpolated as presented in Figure 7 by using proposed C^1 cubic trigonometric curve (a), cubic trigonometric B-spline curves by [12] in (b) and trigonometric cubic Bezier-like curve by [19] in (c).

Table 4: A positive data set by by [2].

i	**0**	**1**	**2**	**3**	**4**
x_i	0	2.0	3.0	7.0	11.0
f_i	0.5	1.5	7.0	9.0	13.0



(a) $m = 1$ (b) $\eta = 2$

(c) $\alpha = 1.5$

Figure 8: Positivity preserving curve using different spline with different values of shape parameter.

Significant shape parameter values have been chosen using the decided fixed range of the shape parameter for interpolating each of the three splines. Based on Figure 8, all the curves generated inherit the genetic characteristic that is the positivity of the data. The presentation of the shape parameter allows the spline generated in giving the visually pleasing and smooth curve. From the two examples given above, in terms of flexibility in choosing the shape parameter values, the proposed cubic trigonometric spline showed greater flexibility since it has a broader range of parameter that is (-2,1] compared to the interval of shape parameter [12], $\eta = (1/2, 2]$ and [19], $\alpha = (0, 2)$.

7 Conclusions

This paper produces a new cubic trigonometric spline with a shape parameter that meets all the required properties. The curve shape can be easily controlled with only a shape parameter without modifying the control points position. The wide range of the shape parameter values in the basis functions gives greater flexibility in interpolating the desired curve. Hence, able to achieve C^1 continuity that preserves the positivity of the data. The developed basis functions are fitted to several sets of data, including opened and closed flower patterns. The generated curves are proven to be visually smooth and pleasant. For future recommendations, this work can be extended to fulfil the geometric continuity, giving better visual smoothness curves and more desirable aesthetics to designs, especially in architecture or car design. The developed basis functions can also be implemented to 3D data such as CT-scan and cloud points.

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Conflicts of Interest The authors declare no conflict of interest.

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